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RAYLEIGH WAVES IN A NONHOMOGENEOUS LAYER RESTING ON A HALF-SPACE

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We study oscillations of Rayleigh wave type, spreading along the surface of a nonhomogeneous (along the depth) elastic layer resting on an elastic half-space. We assume that the Rayleigh velocity on the diurnal surface is less than the transversal velocity in the layer, but larger than the longitudinal velocity in the half-space. In the neighborhood of high frequencies we compute the phase velocity of the wave, similar to the usual Rayleigh wave in the Rayleigh problem for the homogeneous half-space. It is shown that this phase velocity is not real, as a consequence of which the wave dies out as it propagates along the diurnal surface.

1. Formulation of the problem. We consider the following plane problem (Rayleigh's problem): the layer $-\infty < x < +\infty$, $0 < z < \alpha$ $(\alpha > 0)$ is occupied by an isotropic elastic medium with parameters $\lambda(z)$, $\mu(z)$, $\rho(z)$; for $z = \alpha$ the layer is rigidly fixed to a homogeneous isotropic elastic half-space $z > \alpha$, where the parameters λ°, μ°, ρ° are constant. The "diurnal" surface $z = 0$ is free of stress. We study in the domain $z \geqslant 0$ solutions of the system of the elasticity equations of the form

$$\mathbf{u}(x, z, t, k, \sigma) = e^{ik(x-t\sigma)}(-iV_1(z, k, \sigma), 0, V_2(z, k, \sigma)) \qquad (1.1)$$

satisfying the radiation condition at $z \to +\infty$.

We are interested in the eigenvalues of the problem, i.e. such values of $\sigma = \sigma(k, \alpha)$ for which there exist nontrivial solutions $\mathbf{u}(x, z, t, k, \sigma)$. For large values of the wave number k $(k > 0)$ we will study the domain of phase velocities σ which are close to v_R, the Rayleigh velocity at the diurnal surface. We assume that

$$v_p^\circ < v_R < v_m \equiv \min_{0 \leqslant z \leqslant \alpha} v_s(z)$$

Here $v_s(z)$ is the transverse velocity in the layer, v_p° is the longitudinal velocity in in the half-space $z > \alpha$, i.e.

$$v_s(z) = \sqrt{\frac{\mu(z)}{\rho(z)}}, \qquad v_p^\circ = \sqrt{\frac{\lambda^\circ + 2\mu^\circ}{\rho^\circ}}$$

The parameters $\lambda(z)$, $\mu(z)$, $\rho(z)$ are assumed to be twice continuously differentiable functions for $z \in [0, \alpha]$. Thus, in the problem under consideration, the longitudinal and the transverse velocities undergo jumps at the point $z = \alpha$ (Fig. 1). At the surface $z = \alpha$ of the discontinuity of the coefficients, we assume the continuity of the

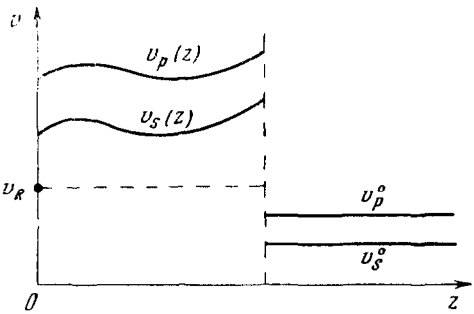

Fig. 1

displacement vector $\mathbf{u}$ and of the normal component τ_z of the stress tensor.

We introduce, as in [1], the four-dimensional vector $\mathbf{Z} = \mathbf{Z}(z, k, \sigma) = (Z_1, Z_2, Z_3, Z_4) = (V_1, V_2, k^{-1}V_1', k^{-1}V_2')$ (the prime denotes differentiation with respect to the variable z) and we obtain the following boundary-value problem:

$$\Omega_0 \mathbf{Z} = 0, \qquad z = 0 \tag{1.2}$$

$$\mathbf{Z}' = (kA(z, \sigma) + B(z))\mathbf{Z}, \quad 0 \leqslant z \leqslant \alpha \tag{1.3}$$

$$\Omega_1 \mathbf{Z} = \Omega_1^\circ \mathbf{Z}^\circ, \qquad z = \alpha \tag{1.4}$$

$$\mathbf{Z}^{\circ\prime} = kA^\circ(\sigma)\mathbf{Z}^\circ, \qquad z \geqslant \alpha \tag{1.5}$$

The vector $\mathbf{Z}^\circ$ must satisfy the radiation condition in the direction $z = +\infty$. Here we have made use of the following notation:

$$\Omega_j = \begin{Vmatrix} 0 & -\mu & \mu & 0 \\ \lambda & 0 & 0 & \nu \\ j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{Vmatrix}, \quad A(z, \sigma) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{m_p^2}{\gamma} & 0 & 0 & \dfrac{1}{\gamma} - 1 \\ 0 & \gamma m_s^2 & \gamma - 1 & 0 \end{Vmatrix}$$

$$B(z) = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \dfrac{\mu'}{\mu} & -\dfrac{\mu'}{\mu} & 0 \\ -\dfrac{\lambda'}{\nu} & 0 & 0 & -\dfrac{\nu'}{\nu} \end{Vmatrix}, \quad A^\circ(\sigma) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{(m_p^\circ)^2}{\gamma^\circ} & 0 & 0 & \dfrac{1}{\gamma^\circ} - 1 \\ 0 & \gamma^\circ (m_s^\circ)^2 & \gamma^\circ - 1 & 0 \end{Vmatrix}$$

$$j = 0,\ 1, \quad \nu = \lambda + 2\mu, \quad \gamma = \mu / \nu, \quad m_l^2 = 1 - \sigma^2 n_l^2, \quad l = p,\ s$$
$$n_p^2 = n_p^2(z) = \rho(z) / \nu(z), \quad n_s^2 = n_s^2(z) = \rho(z) / \mu(z)$$

Here and in the sequel, all the quantities referring to the half-space $z \geqslant \alpha$ are marked with an upper zero index. The relation (1.4) signifies a rigid contact between the layer and the half-space.

The radiation condition is satisfied by the following linear combination of the columns of the fundamental matrix $P^\circ \exp(ik(z - \alpha)\Lambda^\circ)$ of the system (1.5):

$$\mathbf{Z}^\circ|_{z \geqslant \alpha} = P^\circ \exp(ik(z - \alpha)\Lambda^\circ)\varkappa^\circ, \quad \varkappa^\circ = (\varkappa_1^\circ, \varkappa_2^\circ, 0, 0) \tag{1.6}$$

where $\varkappa_1^\circ$, $\varkappa_2^\circ$ are arbitrary constants; the matrix $P^\circ = P^\circ(\sigma)$ reduces $A^\circ(\sigma)$ to the diagonal form $i\Lambda^\circ$, where

$$\Lambda^\circ = \Lambda^\circ(\sigma) = \operatorname{diag}\{M_p, M_s, -M_p, -M_s\}, \quad M_l = M_l(\sigma) = \sqrt{-(m_l^\circ)^2}$$

and $M_l > 0$ for real $\sigma \approx v_R$, $l = p,\ s$.

Let $Y(z, k, \sigma)$ be the fundamental matrix of the system (1.3) on the interval $[0, \alpha]$. The vector $\mathbf{Z}^\circ$ can be extended to the interval $[0, \alpha]$ in the form $\mathbf{Z} = Y(z, k, \sigma)\varkappa$, where $\varkappa = (\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$ is some constant column vector. The contact condition (1.4) gives the relation $\Omega_1(\alpha) Y(\alpha, k, \sigma)\varkappa = \Omega_1^\circ P^\circ(\sigma)\varkappa^\circ$ or

$$\varkappa^\circ = \Phi(\alpha, k, \sigma)\varkappa \tag{1.7}$$
$$\Phi(\alpha, k, \sigma) = (P^\circ(\sigma))^{-1}(\Omega_1^\circ)^{-1}\Omega_1(\alpha)Y(\alpha, k, \sigma)$$

It is convenient to take $\varkappa_1$, $\varkappa_2$ for the two independent arbitrary constants (instead of $\varkappa_1^\circ$, $\varkappa_2^\circ$). Solving the third and the fourth equation of (1.7) relative to $\varkappa_3$, $\varkappa_4$, we obtain

$$\varkappa_3 = b_1\varkappa_1 + b_2\varkappa_2, \quad \varkappa_4 = b_3\varkappa_1 + b_4\varkappa_2 \tag{1.8}$$
$$b_i = b_i(\delta, k), \quad i = 1, 2, 3, 4$$
$$b_1 = (\Phi_{41}\Phi_{34} - \Phi_{31}\Phi_{44}) / q, \quad b_2 = (\Phi_{34}\Phi_{42} - \Phi_{32}\Phi_{44}) / q$$
$$b_3 = (\Phi_{43}\Phi_{31} - \Phi_{33}\Phi_{41}) / q, \quad b_4 = (\Phi_{43}\Phi_{32} - \Phi_{33}\Phi_{42}) / q$$
$$q = \Phi_{33}\Phi_{44} - \Phi_{34}\Phi_{43} \neq 0$$

The boundary condition (1.2) gives the system of four equations

$$\Omega_0(0)\,Y(0, k, \sigma)\,\varkappa = 0 \tag{1.9}$$

from which only the first two equations are nontrivial. Inserting (1.8) into (1.9), we obtain two homogeneous equations for $\varkappa_1$, $\varkappa_2$

$$(F_{j1} + b_1F_{j3} + b_3F_{j4})\,\varkappa_1 + (F_{j2} + b_2F_{j3} + b_4F_{j4})\,\varkappa_2 = 0 \tag{1.10}$$
$$j = 1, 2, \quad F = F(\sigma, k) = \Omega_0(0)\,Y(0, k, \sigma)$$

The eigenvalues of the boundary-value problem (1.2) – (1.5) are determined from the condition of degeneration of the system (1.10).

We formulate the auxiliary "model" Rayleigh problem. Namely, we extend the functions $\lambda(z)$, $\mu(z)$, $\rho(z)$ for $z \geqslant \alpha$ so that for all $z \geqslant 0$ they should be twice continuously differentiable, while starting with some $z = z_* > \alpha$ they should be constant, and for all $z \geqslant 0$ the condition $v_s(z) > v_R$ should hold. Instead of the radiation condition we impose the condition of the decrease of the solution $Z(z, k, \sigma)$ at $z \to \infty$.

In this problem the solution has the form $Y_*(z, k, \sigma)(\varkappa_1, \varkappa_2, 0, 0)$, where $Y_*(z, k, \sigma)$ denotes the fundamental matrix of the model problem; obviously, $Y_*(z, k, \sigma) = Y(z, k, \sigma)$ on the segment $z \in [0, \alpha]$. The boundary condition (1.2) gives the system of four equations $\Omega_0(0)\,Y(0, k, \sigma)(\varkappa_1, \varkappa_2, 0, 0) = 0$, from which the first two are nontrivial

$$F_{j1}\varkappa_1 + F_{j2}\varkappa_2 = 0, \quad j = 1, 2 \tag{1.11}$$

The eigenvalues of the model problem are determined from the degeneration condition of the system (1.11)

$$R(\sigma, k) = 0 \tag{1.12}$$
$$R(\sigma, k) \equiv F_{11}(\sigma, k)\,F_{22}(\sigma, k) - F_{21}(\sigma, k)\,F_{12}(\sigma, k)$$

The system (1.10) differs from (1.11) in the presence of the terms containing b_i ($i = 1, 2, 3, 4$). Further it will be shown that for $k \gg 1$ the quantities b_i are exponentially small, from where we obtain the nearness of the eigenvalues of the initial and the model Rayleigh problems. This phenomenon ought to have been expected sine for short waves the "absorber of energy" (i.e. the underlying half-space $z \geqslant \alpha$) is far away from the diurnal surface in the vicinity of which the energy of Rayleigh waves is essentially concentrated, and the influence of this absorber on the eigenvalue is small.

2. Asymptotic formulas. For the fundamental matrix $Y(z, k, \sigma)$ of the system (1.3) we have an asymptotic formula for $k \to \infty$ (see [2]).

$$Y(z, k, \sigma) = (T(z, \sigma) + O(k^{-1}))\, D(z, \sigma) \exp\left(k \int_0^z \Lambda(\zeta, \sigma)\, d\zeta\right) \qquad (2.1)$$

Here the matrix $T(z, \sigma)$ reduces $A(z, \sigma)$ to the diagonal form

$$\Lambda(z, \sigma) = \operatorname{diag}\{-m_p, -m_s, m_p, m_s\}$$

$$T(z, \sigma) = \begin{Vmatrix} 1 & m_s & 1 & m_s \\ m_p & 1 & -m_p & -1 \\ -m_p & -m_s^2 & m_p & -m_s^2 \\ -m_p^2 & -m_s & -m_p^2 & -m_s \end{Vmatrix}$$

$$D(z, \sigma) = (\rho \operatorname{diag}\{m_p, m_s, m_p, m_s\})^{-1/2}$$

The radicals $m_l(z, \sigma)$ $(l = p, s)$ are determined in the complex plane σ, cut along the rays $\arg(\sigma - v_m) = 0$, $\arg(\sigma + v_m) = \pi$, and those branches are taken for which $m_l(z, 0) > 0$. The asymptotics (2.1) is uniform with respect to $z \in [0, \alpha]$ and $\sigma \in \Sigma$, where Σ is any compactum in the plane σ not containing points of the cuts. As Σ we can take, for example, the circle $|\sigma| \leqslant v_m - \varepsilon$, $\varepsilon > 0$.

Applying formula (2.1) to Eq. (1.12), we obtain the equation

$$R_0(\sigma) + O(k^{-1}) = 0 \qquad (2.2)$$

$$R_0(\sigma) \equiv (1 + m_s^2(0, \sigma))^2 - 4 m_p(0, \sigma) m_s(0, \sigma)$$

Equation (2.2) has a root $\sigma_R(k)$ which is close to the positive root v_R of the equation (*) $R_0(\sigma) = 0$, namely,

$$\sigma_R(k) = v_R + O(k^{-1})$$

In [2] this formula is more precise; it is shown that

$$\sigma_R(k) = v_R + v_1 k^{-1} + O(k^{-2}) \qquad (2.3)$$

and the quantity v_1, depending on $\lambda(0)$, $\mu(0)$, $\rho(0)$, $\lambda'(0)$, $\mu'(0)$, $\rho'(0)$ is computed (as a consequence of the fact that the model problem is self-adjoint, $\operatorname{Im} \sigma_R(k) \equiv 0$).

We proceed now to the computation of the eigenvalues of the initial problem. Substituting the asymptotics (2.1) into the formula (1.8), we obtain the following estimates:

$$b_1 = O(E_p^2), \quad b_2 = O(E_p E_s), \quad b_3 = O(E_p E_s), \quad b_4 = O(E_s^2)$$

$$E_l \equiv \exp\left(-k \int_0^\alpha m_l(z, \sigma)\, dz\right), \qquad l = p, s$$

We take as the domain Σ a neighborhood of the point $\sigma = v_R$. Since $0 < \operatorname{Re} m_s(z, \sigma) < \operatorname{Re} m_p(z, \sigma)$ for $\sigma \in \Sigma$, the principal contribution in the correction to the formula (2.3) is given by $b_4(\sigma, k)$. The equation which gives the eigenvalues of the initial problem takes the form

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*) In the case of a homogeneous half-space, the equation $R_0(\sigma) = 0$ is called the Rayleigh equation, $R(\sigma, k) \equiv R_0(\sigma)$, $0 < v_R < v_s(0)$.

$$R(\sigma, k) + b_4(\sigma, k) S(\sigma, k)[1 + \varepsilon(\sigma, k)] = 0 \tag{2.4}$$

Here

$$S(\sigma, k) \equiv F_{11}(\sigma, k) F_{24}(\sigma, k) - F_{21}(\sigma, k) F_{14}(\sigma, k) = S_0(\sigma) + O(k^{-1})$$

$$S_0(\sigma) \equiv (1 + m_s^2(0, \sigma))^2 + 4 m_p(0, \sigma) m_s(0, \sigma)$$

$$\varepsilon(\sigma, k) = O(E_p / E_s)$$

By virtue of Rouché's theorem, there exists a root of Eq. (2.4) in the neighborhood of the point v_R and

$$\sigma_R(k, \alpha) = \sigma_R(k) + (b_4 S_0 / R_0')\,|_{\sigma=\sigma_R(k)} (1 + o(1)) \tag{2.5}$$

For the imaginary part of $\sigma_R(k, \alpha)$ we have, in particular, the formula

$$\operatorname{Im} \sigma_R(k, \alpha) = -(\beta + O(k^{-1})) \exp\left(-2k \int_0^\alpha \sqrt{1 - n_s^2(z) v_R^2}\, dz\right) \tag{2.6}$$

For the sake of simplicity, we compute the coefficient β in the particular case when the parameters λ and μ are continuous, i.e. when $\lambda(\alpha - 0) = \lambda^\circ$, $\mu(\alpha - 0) = \mu^\circ$; in this case the jump of the velocities at the point $z = \alpha$ takes place only at the expense of the jump of the parameter ρ: $\rho(\alpha - 0) < \rho^\circ$

The result of the computation is

$$\beta = \varphi\psi\eta > 0, \ \eta = S_0(v_R) / R_0'(v_R) = v_R (2 - r)^4 / \{2r[4(1 + \gamma - 2\gamma r) - (2 - r)^3]\} > 0$$

$$\varphi = \exp\left[2 v_1 v_R \int_0^\alpha n_s^2(z)(1 - n_s^2(z) v_R^2)^{-1/2}\, dz\right] > 0$$

$$\psi = \operatorname{Im} b_4(v_R, \infty) = \psi_1 / \psi_2 > 0, \ r = v_R^2 / v_s^2(0)$$

$$\psi_1 = 2\varepsilon_p \varepsilon_s (\xi + \delta_p^2 + \delta_s^2), \ \varepsilon_l = (v_l^0 / v_l(\alpha))^2 < 1$$

$$\delta_l = m_l(\alpha, v_R) / M_l(\alpha, v_R), \ l = p, s$$

$$\xi = (1 - \varepsilon_p)(1 - \varepsilon_s) / [M_p(\alpha, v_R) M_s(\alpha, v_R)] > 0$$

$$\psi_2 = (\xi + \varepsilon_p \varepsilon_s - \delta_p \delta_s)^2 + (\varepsilon_p \delta_s + \delta_p \varepsilon_s)^2$$

Thus, we have obtained the following result: The Rayleigh problem for the system "layer on a half-space" has a solution of the form (1.1), corresponding to the phase velocity $\sigma_R(k, \alpha)$. As a consequence of the fact that $\operatorname{Im} \sigma_R(k, \alpha) < 0$, the Rayleigh wave, running along the surface $z = 0$, has an exponential decay with respect to time.

It should be noted that the first damped Rayleigh waves in a half-space with monotonically decreasing velocities $v_p(z)$, $v_s(z)$ have been studied by Zavadskii [3]. The scalar problem with a jump in the refraction index $n(z)$ (among others with a complex $n(z)$) has been studied in [4].

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